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COMPARISON OF THE VON ZEIPEL AND MODIFIED HANSEN METHODS AS APPLIED TO ARTIFICIAL SATELLITES

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SUMMARY

The solutions to the problem of the near earth satellite without drag obtained by applying the von Zeipel method and the modified Hansen method are compared. Formulas are derived for osculating elements when the modified Hansen theory is expressed in terms of orbital true longitude. Differences in the arbitrary constants are tabulated. Transformations that relate the time element of the two theories are also given.

CONTENTS

COMPARISON OF THE VON ZEIPEL AND MODIFIED HANSEN METHODS AS APPLIED TO ARTIFICIAL SATELLITES

(Manuscript Received May 20,1963)

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INTRODUCTION

Widely different theories are often used in computing orbits of artificial satellites. It is of interest to examine the results of different theories when they are applied to the basic problem of the near earth satellite without drag. Of special importance are the major theories of celestial mechanics introduced by Brouwer[1] and by Musen[2,3] in solving this problem.

Brouwer[1] applied the method of von Zeipel to the near earth satellite problem and obtained analytic representations for the osculating Delaunay and Keplerian elements. The results are given by Brouwer to order J_2 in the elements and J_2^2 in the mean motions, where J_2 is the coefficient of the second zonal harmonic of the earth's potential, and equals the quantity $2k_2$ appearing in the articles of Brouwer and Musen. Musen[2,3], on the other hand, first modified Hansen's method, then by applying it to the same problem of the near earth satellite without drag, showed how to obtain the position of the satellite in a semi-analytic manner to any prescribed order of J_2. The solution of the satellite problem in terms of orbital true longitude by Musen[3] is considered below.

The results obtained by Brouwer are given in a form convenient for comparison with the results of many authors. Indeed Kozai[4], Garfinkel[5] and others have readily compared their solutions with Brouwer's. However, since Musen's formulations of the problem are intended to provide numerical results of high precision for the position of a satellite, explicit analytic formulations of the perturbations of the elements do not appear in his articles. For that reason, formulas are given here for elements derived from the modified Hansen theory in terms of orbital true longitude so that the results of Musen and Brouwer can be compared.

As would be expected, the differences between the two theories are exhibited in the respective choices of the arbitrary constants and in the arguments of the trigonometric terms. The constants of both theories are discussed and presented in tabular form. The transformations of the variables of the angular arguments are presented. Therefore, when the solutions to the satellite problem are carried out to the same order in J_2 by the methods of Brouwer and Musen, full correspondence can be obtained by taking into account the differences in the constants and the angular variables.

THE OSCULATING ELEMENTS

The definitions of the osculating elements appearing in Brouwer's article may be found in any text on celestial mechanics—for example, Brouwer and Clemence[6]. It is, moreover, a relatively simple matter to find expressions for the osculating elements of the modified Hansen theory as expressed in terms of orbital true longitude; these differ from the corresponding formulas of the modified Hansen theory as expressed in terms of eccentric anomaly given by Bailie and Bryant[7], since the W functions differ slightly. To indicate how such representations of osculating elements are derived, we shall now review briefly some of the concepts and definitions of the modified Hansen theory as expressed in terms of orbital true longitude.

DEFINITIONS FROM THE MODIFIED HANSEN THEORY

When the differential equations given in Musen's articles[2,3] are solved, expressions for the components Ξ, Υ, and Ψ, of the $\bar{W}$ function, for the λ parameters, and for the perturbation of the pseudo-time $n_0\,\delta z$ result. The functions Ξ, Υ, and Ψ are expressed in terms of orbital true longitude and are related to osculating elements by the formulas

$$\left.\begin{aligned}
\Xi &= -1 - \frac{h_0}{h} + 2\,\frac{h}{h_0} \,, \\
\Upsilon &= 2\,\frac{h}{h_0}\, e\cos\phi - \left(1 + \frac{h_0}{h}\right)e_0 \,, \\
\Psi &= 2\,\frac{h}{h_0}\, e\sin\phi \,.
\end{aligned}\right\} \tag{1}$$

Here $-\phi$ is the deviation of the osculating true anomaly from the true anomaly of the auxiliary ellipse, e is the osculating eccentricity, and h is proportional to the reciprocal of the Delaunay variable G; that is,

$$G = \frac{\mu}{h} \,. \tag{2}$$

The quantities h_0 and e_0 are the elements of Hansen's auxiliary ellipse.

The λ parameters are defined by the formulas

$$\left.\begin{aligned}
\lambda_1 &= \sin\frac{i}{2}\cos N & \qquad \lambda_3 &= \cos\frac{i}{2}\sin K \\
\lambda_2 &= \sin\frac{i}{2}\sin N & \qquad \lambda_4 &= \cos\frac{i}{2}\cos K.
\end{aligned}\right\} \tag{3}$$

Here i is the osculating angle of inclination of the orbit plane and corresponds to I in Brouwer's development. The quantities K and N are Fourier series of the order of the perturbations and do not contain secular terms.

The angular variables are given by the formulas

$$\left.\begin{aligned} f &= cv - \pi_0 - \phi \ , \\ \omega &= (g - c)v + \left(\pi_0 - \theta_0\right) + \phi + K + N \ , \\ \theta &= (1 - h')v + \theta_0 + K - N \ . \end{aligned}\right\} \tag{4}$$

The quantities f, ω, and θ are the osculating true anomaly, argument of perigee, and longitude of the node, respectively. The quantities g, c, and h' in the right hand side of Equations 4 are proportional to the mean motions of the argument of latitude, mean anomaly, and the longitude of the ascending node, respectively. The quantities π_0 and θ_0 are prescribed constants.

The time element of the auxiliary ellipse is denoted by the symbol z and is often called the pseudo-time. When orbital true longitude is the argument, the mean anomaly of the auxiliary ellipse is $c\left(n_0\right)_H z$. The symbol n_0 appears with different meanings in the articles of Brouwer and Musen; therefore the symbol $\left(n_0\right)_H$ is adopted here instead of the n_0 appearing in Musen's article. The quantity δz is the deviation of the pseudo-time from the unperturbed satellite time.

OSCULATING ELEMENTS FOR THE MODIFIED HANSEN THEORY

By inverting Equations 1 and 3 it is readily found that

$$\left.\begin{aligned} G &= \frac{\mu}{h} = \frac{\mu}{h_0}\left(1 - \frac{\Xi}{3} + \frac{2}{27}\Xi^2 + \dots\right) \ , \\ e &= e_0 + \frac{1}{2}\left(\Upsilon - e_0\Xi\right) + \frac{1}{24}\left(4e_0\Xi^2 - 4\Xi\Upsilon + \frac{3\Psi^2}{e_0}\right) + \dots \ , \\ \frac{H}{G} &= \cos i = \cos i_0\left(1 + \frac{\Xi}{3} + \frac{\Xi^2}{27} + \dots\right) \ . \end{aligned}\right\} \tag{5}$$

Similarly, the quantities associated with the angular variables are found to be

$$\left.\begin{aligned} \phi &= \frac{\Psi}{2e_0} + \frac{\Psi}{4e_0}\left[\frac{\Xi}{3} - \frac{\Upsilon}{e_0}\right] + \dots \ , \\ K + N &= \frac{\lambda_3}{\cos\frac{i_0}{2}} + \frac{\lambda_2}{\sin\frac{i_0}{2}} - \frac{\Xi}{12}\cos i_0\left[\frac{\lambda_3}{\cos^3\frac{i_0}{2}} - \frac{\lambda_2}{\sin^3\frac{i_0}{2}}\right] + \dots \ , \\ K - N &= \frac{\lambda_3}{\cos\frac{i_0}{2}} - \frac{\lambda_2}{\sin\frac{i_0}{2}} - \frac{\Xi}{12}\cos i_0\left[\frac{\lambda_3}{\cos^3\frac{i_0}{2}} + \frac{\lambda_2}{\sin^3\frac{i_0}{2}}\right] + \dots \ . \end{aligned}\right\} \tag{6}$$

It is instructive to derive an expression for the perturbation of the radius vector of the satellite using Equations 5 and 6. This leads to an important result already given in the modified Hansen theory. If u denotes the reciprocal of the radius vector of the satellite, and δ the deviation of an osculating element from its value in the auxiliary ellipse, then

$$\delta u = u - \bar{u} = h_0 \frac{\partial u}{\partial h} \delta\left(\frac{h}{h_0}\right) + \frac{\partial u}{\partial e} \delta e + \frac{\partial u}{\partial f} \delta f \quad . \tag{7}$$

But we have

$$\left.\begin{aligned} u &= \frac{h^2}{\mu}(1 + e \cos f) \quad , \\ \frac{\partial u}{\partial h} &= 2\frac{h}{\mu}(1 + e \cos f) \quad , \\ \frac{\partial u}{\partial e} &= \frac{h^2}{\mu} \cos f \quad , \\ \frac{\partial u}{\partial f} &= -\frac{eh^2}{\mu} \sin f \quad , \\ \bar{u} &= \frac{h_0^2}{\mu}(1 + e_0 \cos \bar{f}) \quad , \end{aligned}\right\} \tag{8}$$

where $f = \bar{f} - \phi$; and from Equations 5 and 6 we have, to order J_2,

$$\left.\begin{aligned} \delta\left(\frac{h}{h_0}\right) &= \frac{\Xi}{3} \quad , \\ \delta e &= \frac{\Upsilon - e_0 \Xi}{2} \quad , \\ \delta f &= -\phi = -\frac{\Psi}{2e_0} \quad . \end{aligned}\right\} \tag{9}$$

Substituting the required quantities from Equations 8 into Equation 7 we get, to order J_2,

$$\delta u = \frac{1}{2} \frac{h_0^2}{\mu} \bar{W} + \frac{\Xi}{6} \bar{u} \quad , \tag{10}$$

where $\bar{W} = \Xi + \Upsilon \cos \bar{f} + \Psi \sin \bar{f}$, which is consistent with the results of the modified Hansen theory.

COMPARISON OF RESULTS TO THE FIRST ORDER IN J_2

By solving the equations given by Musen, first order analytic solutions for the quantities Ξ, Υ, Ψ, and the λ parameters were obtained by Bailie and Fisher[8]. When the analytic expressions for Ξ,

Υ, and Ψ are substituted into Equations 5 immediate agreement is obtained with the periodic part of the elements G, e, and I obtained in Brouwer's solution. Similarly, agreement for the periodic part of the expressions for the angular variables ω and θ given by Equation 4 with the variables g and h can be readily obtained, when the analytic results of Bailie and Fisher are introduced.

It has now been indicated that the periodic part of the solution of the elements of the satellite problem by Brouwer and Musen agree to the first order in J_2. Although differences in the arbitrary constants and arguments of the trigonometric terms do exist, they do not appear in the first order solutions for the trigonometric parts of the elements since they have J_2 as a multiplier. These differences are exhibited in the terms of the second order and are discussed below.

THE ARBITRARY CONSTANTS OF THE THEORIES

Differences of order J_2 appear in the arbitrary constants of the solutions of the satellite problem by Brouwer and by Musen. The quantities c_0 and $c_1 \cos f$ in Musen's work are added to the $\bar{W}$ function and consequently the constants c_0 and c_1 are added to Ξ and Υ; These constants thus occur in the solution for those elements derived from Ξ and Υ. Constants also appear in the solution for the elements by Brouwer. In order to compare the two theories the constants of the elements G, e, and $\cos i$ with respect to true anomaly are found from Brouwer's development to order J_2. Similarly, constants with respect to orbital true longitude are found in the article of Bailie and Fisher[8] and are listed in Table 1.

Table 1

Constants Appearing in the Satellite Theories (Order J_2.)

Quantity	Brouwer's Notation (von Zeipel method)	Musen's Notation (modified Hansen method)
$G = \frac{\mu}{h}$	G''	$\frac{\mu}{h_0}\left(1 - \frac{c_0}{3}\right)$
e	$e'' - \frac{\mu^2 J_2 \left(1 - 3\cos^2 I''\right)}{8e''G''^4}\left(5 - 3\eta''^2 - 2\eta''^3\right)$	$e_0 + \frac{c_1 - e_0 c_0}{2}$
$\frac{H}{G} = \cos i$	$\cos I''$	$\cos i_0\left(1 + \frac{c_0}{3}\right)$
mean motion of mean anomaly	$\frac{dl''}{dt}$	$c\,(n_0)_H$

The constants appearing in Table 1 are defined as follows:

$$\left.\begin{aligned}
\eta'' &= \sqrt{1 - e''^2} \quad , \\
\frac{dl''}{dt} &= n_0 - \frac{3}{4}\frac{n_0\mu^2\, J_2}{L'\, G''^3}\left(1 - 3\cos^2 I''\right) \quad , \\
c &= 1 + \frac{3}{4} J_2 \frac{h_0^4}{\mu^2}\left(1 - 3\cos^2 i_0\right) \, , \\
c_0 &= \frac{3}{4} J_2 \frac{h_0^4}{\mu^2}\left(1 - 3\cos^2 i_0\right)\left(4 - 2\sqrt{1 - e_0^2}\right) \, , \\
c_1 &= \frac{1}{4}\frac{J_2}{e_0}\frac{h_0^4}{\mu^2}\left(1 - 3\cos^2 i_0\right)\left[4\left(1 - \sqrt{1 - e_0^2}\right) + 3e_0^2 - 2e_0^2\sqrt{1 - e_0^2}\right] \, .
\end{aligned}\right\} \qquad (11)$$

These values are taken from the article of Brouwer and from the article of Bailie and Fisher[8].

The relations between the mean motions of the argument of perigee and the longitude of the node in the articles of Brouwer and Musen are given by the formulas

$$\left.\begin{aligned}
\frac{dg''}{dt} &= \left(n_0\right)_H (g - c) \quad , \\
\frac{dh''}{dt} &= \left(n_0\right)_H (1 - h') \quad .
\end{aligned}\right\} \qquad (12)$$

Formulas to order J_2^2 for these mean motions are given in the article of Brouwer and the article of Bailie and Fisher. At first sight the terms in J_2^2 seem to disagree. However, by taking the relationships given in Table 1 into account, full agreement is obtained to order J_2^2 in the mean motion of the variables as defined in Equations 12.

The differences in the constants given in Table 1 will also be exhibited in the coefficients of trigonometric terms of order J_2^2 in the elements derived by the methods of Brouwer and of Musen. Additional differences appear in these coefficients and are due to differences in the *arguments* of the trigonometric terms. We shall now describe these.

THE TIME ELEMENTS OF THE THEORIES

In the von Zeipel method adopted by Brouwer the true anomalies f and f' appear; Brouwer then shows how to relate these true anomalies to the true time of the satellite. In the Hansen method modified by Musen the true anomaly of the auxiliary ellipse $\bar{f}$ (or ξ as it is denoted by Bailie and Fisher[8]) appears; Musen shows how to relate $\bar{f}$ to the true time. The true anomalies in the two theories differ by trigonometric terms of order J_2; consequently, it is logical to apply Taylor's theorem to find the relation between these two true anomalies.

We recall that the true anomaly is a function of the eccentricity as well as of the mean anomaly, as is shown by the equation of the center.[6] Also, we have

$$\left.\begin{aligned} f_{osc} &= \bar{f} - \phi \ , \\ e_{osc} &= e_0 + \delta e \ , \end{aligned}\right\} \tag{13}$$

where f_{osc} and e_{osc} are the osculating true anomaly and osculating eccentricity, while ϕ and δe are of order J_2.

The quantity f appearing in Brouwer's article is related to the osculating mean anomaly l by the equation

$$\frac{df}{dl} = \left(1 - e_0^2\right)^{\frac{-3}{2}} \left(1 + e_0 \cos f\right)^2 \ . \tag{14}$$

We then find by Taylor's theorem that

$$F(\bar{f}) = F(f) + \left(\phi + \frac{\partial f}{\partial e}\,\delta e\right) \frac{\partial F}{\partial \bar{f}} \ , \tag{15}$$

where

$$\frac{\partial f}{\partial e} = \left(\frac{2 + e_0 \cos f}{1 - e_0^2}\right) \sin f \ . \tag{16}$$

Equation 15 transforms a function of the true anomaly of the auxiliary ellipse, $\bar{f}$, to a function of the true anomaly f appearing in Brouwer's article. To extend this transformation so that a function of $\bar{f}$ may be expressed in terms of the mean true anomaly f', we simply apply Taylor's Theorem again to obtain

$$F(\bar{f}) = F(f') + \left(\phi + \frac{\partial f}{\partial e}\,\delta e + \frac{df}{dl}\,\Delta l\right) \frac{\partial F}{\partial \bar{f}} \ , \tag{17}$$

to the first order in J_2. Here f' is the mean true anomaly in the sense given in Brouwer's article and may be evaluated by Kepler's equation for a given instant of time. The perturbation Δl is the deviation of the mean anomaly from its mean value and is given by the formula $\Delta l = -\partial S_1/\partial L'$ in Brouwer's article. It may also be found from the variation equation in terms of orbital true longitude by the methods adopted in the article of Bailie and Fisher.

In particular, if $F(\bar{f}) = \sin \bar{f}$, we have

$$\sin \bar{f} = \sin f' + \left(\phi + \frac{\partial f}{\partial e}\,\delta e + \frac{df}{dl}\,\Delta l\right) \cos \bar{f} \ . \tag{18}$$

The multiplier of $\cos \bar{f}$ is of order J_2, so that when f' is given $\bar{f}$ may be found by successive approximations.

It is possible to arrive at Equation 17 by approaching the transformation from a somewhat different point of view. From the work of E. Brown[9] we have

$$\delta f = \frac{df}{dl}\,\delta l + \frac{\partial f}{\partial e}\,\delta e \quad . \tag{19}$$

Since the symbol δ refers to the deviation of the osculating element from the corresponding element of the auxiliary ellipse, we have

$$\delta l = l - \left[c\left(n_0\right)_H z + l_0''\right] \quad , \tag{20}$$

where the quantity in brackets is the mean anomaly of the auxiliary ellipse. Since by Table 1

$$c\left(n_0\right)_H = \frac{dl''}{dt} \tag{21}$$

and

$$z - t = \delta z \quad ,$$

we have, from Equation 20,

$$\delta l = \left[l - \left(\frac{dl''}{dt}\,t + l_0''\right)\right] - c\left(n_0\right)_H \delta z \quad . \tag{22}$$

When only the short period terms of the mean anomaly are considered, we have

$$\delta l = \Delta l - c\left(n_0\right)_H \delta z \quad . \tag{23}$$

Or, substituting in Equation 19, we find

$$\frac{df}{dl}\,c\left(n_0\right)_H \delta z = \phi + \frac{\partial f}{\partial e}\,\delta e + \frac{df}{dl}\,\Delta l \quad , \tag{24}$$

which equals the multiplier of $\partial F/\partial \bar{f}$ in Equation 17. Consequently, Equation 17 may be thought of as transforming a function of the pseudo-time z to a function of time t (to order J_2) by the relation

$$F(z) = F(t) + \frac{\partial F}{\partial z}\,\delta z \quad . \tag{25}$$

In order to complete the transformation, the argument of perigee is taken into consideration. From Equations 4 it is seen that a term proportional to the equation of the center must be included in the transformations. It then follows that

$$F(\bar{f},\bar{\omega}) = F(f',g') + \left(\phi + \frac{\partial f}{\partial e}\,\delta e + \frac{df}{dl}\,\Delta l\right)\frac{\partial F}{\partial \bar{f}} + (g-c)(f-l)\,\frac{\partial F}{\partial \bar{\omega}} \quad , \tag{26}$$

where

$$\bar{\omega} = (g-c)v + \left(\pi_0 - \theta_0\right) \quad .$$

If $F(f')$ and $F(f',g')$ in the right-hand sides of Equations 17 and 26 represent periodic terms of order J_2, then the subsequent terms in the right-hand sides of these equations will be of order J_2^2. Thus, by means of Equations 17 and 26, it is possible to derive comparisons of the periodic terms of the two theories when they are both developed to order J_2^2.

Kozai[10] has extended the work of Brouwer[1] to order J_2^2 in the periodic terms and J_2^3 in the secular terms. Unfortunately, a corresponding extension has not been made of the analytic results of Bailie and Fisher[8]. However, for the purpose of checking the formulas of the present paper the author has obtained the development of only the short period terms of the Delaunay variable G to order J_2^2 in terms of orbital true longitude. This was done so that comparison could be made with the corresponding terms obtained by Kozai using the method of von Zeipel. For this variable, agreement was obtained.

SUMMARY AND CONCLUSIONS

The solutions to the problem of the near earth satellite without drag given by Brouwer[1] and by Musen[2,3] agree when carried out to the same order in J_2. Due allowance must be made for the differences in the constants and in the ways of expressing the time element.

The differences in the arbitrary constants have been tabulated here to the first order in J_2, and transformations have been given relating the true anomaly of the auxiliary ellipse to that of the satellite.

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Appendix A

Analytic Results from the Modified Hansen Theory

By solving the equations given by Musen*, first order analytic solutions for the quantities Ξ, Υ, Ψ, λ_2, λ_3, g, $1-h'$, and $g-c$ were obtained by Bailie and Fisher†. These are listed here for the convenience of the reader.

To conform to modern notation, the quantities k_2, k_3, and k_4 appearing in the modified Hansen theory are here designated by $J_2/2$, $J_3/2$, and $-J_4/8$ respectively. In formulas A1 through A5 the quantities g and c introduced by integration have been restored, along with the quantity μ; these were set equal to unity by Bailie and Fisher who were then concerned only with terms of order J_2.

$$\Xi = \frac{3}{2} J_2 \frac{h_0^4}{\mu^2} \left(1 - 3\cos^2 i_0\right)\left[2 - \sqrt{1 - e_0^2}\right]$$

$$- \frac{9}{4} J_2 \frac{h_0^4}{\mu^2}\left(1 - \cos^2 i_0\right)\left\{\frac{\cos 2\eta}{g} + e_0 \frac{\cos(\xi + 2\eta)}{c + 2g} + e_0 \frac{\cos(\xi - 2\eta)}{2g - c}\right.$$

$$\left. - \frac{1}{12}\left[e_0^2 - \frac{10\, e_0^2 \cos^2 i_0}{1 - 5\cos^2 i_0}\right]\cos(2\xi - 2\eta)\right\} - \frac{3}{2}\frac{J_3}{J_2}\frac{h_0^2}{\mu} e_0 \sin i_0 \sin(\xi - \eta)$$

$$+ \frac{15}{16}\frac{J_4}{J_2}\frac{h_0^4}{\mu^2} e_0^2\left[1 - 3\cos^2 i_0 - \frac{8\cos^4 i_0}{1 - 5\cos^2 i_0}\right]\cos(2\xi - 2\eta) \;; \qquad \text{(A1)}$$

$$\Upsilon = \frac{1}{8} J_2 \frac{h_0^4}{\mu^2}\left(1 - 3\cos^2 i_0\right)\left[2e_0 + \frac{8e_0 + 4e_0^3}{1 + \sqrt{1 - e_0^2}}\right] - \frac{1}{8}\frac{J_2}{c}\frac{h_0^4}{\mu^2}\left(1 - 3\cos^2 i_0\right)\left[\left(12 + 3e_0^2\right)\cos\xi\right.$$

$$\left. + 6e_0 \cos 2\xi + e_0^2 \cos 3\xi\right] + \frac{3}{16} J_2 \frac{h_0^4}{\mu^2}\left(1 - \cos^2 i_0\right)\left\{8e_0 \frac{\cos 2\eta}{g}\right.$$

*Musen, P., "Application of Hansen's Theory to the Motion of an Artificial Satellite in the Gravitational Field of the Earth," *J. Geophys. Res.* 64(12):2271-2279, December 1959.

†Bailie, A., and Fisher, D., "An Analytic Representation of Musen's Theory of Artificial Satellites in Terms of the Orbital True Longitude," NASA Technical Note D-1468, January 1963

$$+ \left(28 + 5e_0^2\right) \frac{\cos\,(\xi + 2\eta)}{c + 2g} + \left(4 - e_0^2\right) \frac{\cos\,(\xi - 2\eta)}{2g - c} + 12e_0 \frac{\cos\,(2\xi + 2\eta)}{c + g}$$

$$+ \frac{1}{3}\left[20e_0 + e_0^3 - \frac{\left(20e_0 + 10e_0^3\right)\cos^2 i_0}{1 - 5\cos^2 i_0}\right]\cos(2\xi - 2\eta) + 5e_0^2 \frac{\cos\,(3\xi + 2\eta)}{3c + 2g}$$

$$\left. + e_0^2 \frac{\cos\,(3\xi - 2\eta)}{3c - 2g}\right\} - \frac{1}{2}\frac{J_3}{J_2}\frac{h_0^2}{\mu}\left(2 + e_0^2\right)\sin i_0 \sin\,(\xi - \eta)$$

$$+ \frac{5}{16}\frac{J_4}{J_2}\frac{h_0^4}{\mu^2}\left(2e_0 + e_0^3\right)\left[1 - 3\cos^2 i_0 - \frac{8\cos^4 i_0}{1 - 5\cos^2 i_0}\right]\cos\,(2\xi - 2\eta) \quad ; \tag{A2}$$

$$\Psi = -\frac{1}{8}\frac{J_2}{c}\frac{h_0^4}{\mu^2}\left(1 - 3\cos^2 i_0\right)\left[\left(12 + 9e_0^2\right)\sin\xi + 6e_0 \sin 2\xi + e_0^2 \sin 3\xi\right]$$

$$+ \frac{3}{16} J_2 \frac{h_0^4}{\mu^2}\left(1 - \cos^2 i_0\right)\left\{12e_0 \frac{\sin 2\eta}{g} + \left(28 + 11e_0^2\right)\frac{\sin\,(\xi + 2\eta)}{c + 2g}\right.$$

$$+ \left(4 - 7e_0^2\right)\frac{\sin\,(\xi - 2\eta)}{2g - c} + 12e_0 \frac{\sin\,(2\xi + 2\eta)}{c + g}$$

$$+ \frac{1}{3}\left[20e_0 + e_0^3 - 20e_0\cos^2 i_0 - \frac{\left(100e_0 - 20e_0^3\right)\cos^4 i_0}{1 - 5\cos^2 i_0} + \frac{100e_0^3 \cos^6 i_0}{\left(1 - 5\cos^2 i_0\right)^2}\right]\sin\,(2\xi - 2\eta)$$

$$\left. + 5e_0^2 \frac{\sin\,(3\xi + 2\eta)}{3c + 2g} + e_0^2 \frac{\sin\,(3\xi - 2\eta)}{3c - 2g}\right\}$$

$$+ \frac{J_3}{J_2}\frac{h_0^2}{\mu}\sin i_0\left[1 - \frac{2e_0^2 \cos^2 i_0}{1 - 5\cos^2 i_0}\right]\cos\,(\xi - \eta)$$

$$+ \frac{5}{16}\frac{J_4}{J_2}\frac{h_0^4}{\mu^2}\left[2e_0 + e_0^3 - \left(6e_0 + 5e_0^3\right)\cos^2 i_0\right.$$

$$\left. - \frac{\left(16e_0 + 12e_0^3\right)\cos^4 i_0}{1 - 5\cos^2 i_0} + \frac{16e_0^3 \cos^6 i_0}{\left(1 - 5\cos^2 i_0\right)^2}\right]\sin\,(2\xi - 2\eta) \quad ; \tag{A3}$$

$$\lambda_2 = \frac{3}{8} J_2 \frac{h_0^4}{\mu^2}\sin i_0 \cos i_0 \cos\frac{i_0}{2}\left\{2e_0 \frac{\sin\xi}{c} - \frac{\sin 2\eta}{g} - e_0 \frac{\sin\,(\xi + 2\eta)}{c + 2g}\right.$$

$$+ e_0 \frac{\sin\,(\xi - 2\eta)}{2g - c} + \left[-\frac{e_0^2\left(8 + 3\cos i_0 - 18\cos^2 i_0\right)}{6\left(1 - 5\cos^2 i_0\right)}\right.$$

$$+ \frac{e_0^2 \left(5 + \cos i_0 - 6 \cos^2 i_0\right) \left(1 - 15 \cos^2 i_0\right)}{12 \left(1 - 5 \cos^2 i_0\right)^2} \Bigg] \sin (2\xi - 2\eta) \Bigg\}$$

$$- \frac{1}{4} \frac{J_3}{J_2} \frac{h_0^2}{\mu} e_0 \cos i_0 \cos \frac{i_0}{2} \left(\frac{1 - 2 \cos i_0 - 3 \cos^2 i_0}{1 - 5 \cos^2 i_0} \right) \cos (\xi - \eta)$$

$$- \frac{5}{16} \frac{J_4}{J_2} \frac{h_0^4}{\mu^2} e_0^2 \sin i_0 \cos i_0 \cos \frac{i_0}{2} \Bigg[\frac{4 - 7 \cos^2 i_0}{1 - 5 \cos^2 i_0}$$

$$- \frac{\left(5 + \cos i_0 - 6 \cos^2 i_0\right) \left(1 - 7 \cos^2 i_0\right)}{2 \left(1 - 5 \cos^2 i_0\right)^2} \Bigg] \sin (2\xi - 2\eta) \quad ; \qquad \textbf{(A4)}$$

$$\lambda_3 = - \frac{3}{8} J_2 \frac{h_0^4}{\mu^2} \sin i_0 \cos i_0 \sin \frac{i_0}{2} \Bigg\{ 2e_0 \frac{\sin \xi}{c} - \frac{\sin 2\eta}{g} - e_0 \frac{\sin (\xi + 2\eta)}{c + 2g}$$

$$+ e_0 \frac{\sin (\xi - 2\eta)}{2g - c} + \Bigg[- \frac{e_0^2 \left(8 - 3 \cos i_0 - 18 \cos^2 i_0\right)}{6 \left(1 - 5 \cos^2 i_0\right)}$$

$$+ \frac{e_0^2 \left(5 - \cos i_0 - 6 \cos^2 i_0\right) \left(1 - 15 \cos^2 i_0\right)}{12 \left(1 - 5 \cos^2 i_0\right)^2} \Bigg] \sin (2\xi - 2\eta) \Bigg\}$$

$$+ \frac{1}{4} \frac{J_3}{J_2} \frac{h_0^2}{\mu} e_0 \cos i_0 \sin \frac{i_0}{2} \left(\frac{1 + 2 \cos i_0 - 3 \cos^2 i_0}{1 - 5 \cos^2 i_0} \right) \cos (\xi - \eta)$$

$$+ \frac{5}{16} \frac{J_4}{J_2} \frac{h_0^4}{\mu^2} e_0^2 \sin i_0 \cos i_0 \sin \frac{i_0}{2} \Bigg[\frac{4 - 7 \cos^2 i_0}{1 - 5 \cos^2 i_0}$$

$$- \frac{\left(5 - \cos i_0 - 6 \cos^2 i_0\right) \left(1 - 7 \cos^2 i_0\right)}{2 \left(1 - 5 \cos^2 i_0\right)^2} \Bigg] \sin (2\xi - 2\eta) \quad ; \qquad \textbf{(A5)}$$

$$g = 1 + \frac{3}{2} J_2 \frac{h_0^4}{\mu^2} \cos^2 i_0 + \frac{15}{32} J_4 \frac{h_0^8}{\mu^4} \left(2 + 3e_0^2\right) \left(3 \cos^2 i_0 - 7 \cos^4 i_0\right)$$

$$+ \frac{3}{32} J_2^2 \frac{h_0^8}{\mu^4} \left[\left(110 - 48 \sqrt{1 - e_0^2} + 13e_0^2\right) \cos^2 i_0 - \left(290 - 144 \sqrt{1 - e_0^2} + 9e_0^2\right) \cos^4 i_0 \right] ; \qquad \textbf{(A6)}$$

$$
l - h' = -\frac{3}{2} J_2 \frac{h_0^4}{\mu^2} \cos i_0 - \frac{15}{32} J_4 \frac{h_0^8}{\mu^4} \left(2 + 3e_0^2\right) \left(3 \cos i_0 - 7 \cos^3 i_0\right)
$$

$$
- \frac{3}{32} J_2^2 \frac{h_0^8}{\mu^4} \left[\left(88 - 40 \sqrt{1 - e_0^2} + 9e_0^2\right) \cos i_0 - \left(236 - 120 \sqrt{1 - e_0^2} + 5e_0^2\right) \cos^3 i_0\right]; \quad \text{(A7)}
$$

$$
g - c = -\frac{3}{4} J_2 \frac{h_0^4}{\mu^2} \left(1 - 5 \cos^2 i_0\right) - \frac{15}{128} J_4 \frac{h_0^8}{\mu^4} \left[\left(12 + 9e_0^2\right) - \left(144 + 126e_0^2\right) \cos^2 i_0 \right.
$$

$$
\left. + \left(196 + 189e_0^2\right) \cos^4 i_0\right]
$$

$$
- \frac{3}{128} J_2^2 \frac{h_0^8}{\mu^4} \left[\left(162 - 64 \sqrt{1 - e_0^2} + 25e_0^2\right) - \left(1500 - 672 \sqrt{1 - e_0^2} + 126e_0^2\right) \cos^2 i_0 \right.
$$

$$
\left. + \left(2810 - 1440 \sqrt{1 - e_0^2} + 45e_0^2\right) \cos^4 i_0\right], \quad \text{(A8)}
$$

where

$$
\xi = cv - \pi_0 \text{ and } \eta = gv - \theta_0 .
$$

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